Classic Inequalities

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Definition 1 *A real-valued function f on an interval $[a, b]$ is called* convex *if, for any points $x, y \in [a, b]$ the following inequality holds for all $0 < \lambda < 1$:*

$$\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y).$$

If the inequality holds in the reverse direction, f is concave.

Many criteria exist for proving convexity. Graphically, f is convex iff the set of points $\{(x, y) \in [a, b] \times \mathbb{R} | y \geq f(x)\}$ is convex (the line segment joining any points of the set lies entirely within the set). If f is continuous, then it suffices to check the statement for $\lambda = \frac{1}{2}$ (or any other fixed $0 < \lambda < 1$). (Conversely, if f is convex on I, then f is continuous except possibly at the endpoints of I.) If f is continuous on $[a, b]$ and twice differentiable on (a, b), then it suffices to check that $f''(x) \geq 0$ for $x \in (a, b)$.

There are many useful convex and concave functions. For example, e^x is a convex function of x, as is x^c for $c \geq 1$ or $c < 0$. Common concave functions include $\ln(x)$ and x^c for $0 < c \leq 1$.

Definition 2 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be two vectors of real numbers, and suppose that $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ are their components in decreasing order. Then $\mathbf{x}$* majorizes *$\mathbf{y}$, denoted $\mathbf{x} \succ \mathbf{y}$, if*

$$x_1 + \cdots + x_i \geq y_1 + \cdots + y_i \quad \text{for all} \quad 1 \leq i \leq n - 1,$$

and $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

Equality of the full sums is an important requirement in majorization.

Theorem 1 (Karamata) *Let f be a convex function on an interval I and suppose that the reals $x_1, \ldots, x_n \in I$ majorize the reals $y_1, \ldots, y_n \in I$. Then*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n)$$

Theorem 2 (Jensen) *Let f be a convex function on an interval I. Then for any reals $x_1, \ldots, x_n \in I$,*

$$f(x_1) + \cdots + f(x_n) \geq f\left(\frac{x_1 + \cdots + x_n}{n}\right).$$

If f is concave, then the inequality is flipped.

Theorem 3 (AM-GM) *Let $a_1, \ldots, a_n$ be positive reals. Then*

$$a_1 + \cdots + a_n \geq n \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 4 (Hölder) *Let $a_1, \ldots, a_n; b_1, \ldots, b_n; \cdots; z_1, \ldots, z_n$ be sequences of nonnegative real numbers, and let $\lambda_a, \lambda_b, \ldots, \lambda_z$ positive reals which sum to 1. Then*

$$(a_1 + \cdots + a_n)^{\lambda_a} (b_1 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_z} b_n^{\lambda_b} \cdots z_n^{\lambda_z}$$

Cauchy's inequality is a special case of Hölder's inequality where $\lambda_a = \lambda_b = 1/2$.

Theorem 5 (Power Mean) *Let r and s be nonzero-real numbers with $r > s$. Then for positive reals $a_1, \dots, a_n$,*

$$\left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{a_1^s + \cdots + a_n^s}{n}\right)^{\frac{1}{s}}.$$

Moreover, if we interpret the 0th power-mean as the geometric mean, the statement is valid for all reals r and s. Observe that the famous AM-GM-HM inequality is a special case of the power-mean inequality.

Theorem 6 (Rearrangement) *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be two nonincreasing sequences of real numbers. Then, for any permutation π of $\{1, 2, \dots, n\}$, we have*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{\pi(1)} + a_2b_{\pi(2)} + \cdots + a_nb_{\pi(n)} \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1.$$

Theorem 7 (Chebyshev) *Let $a_1 \geq a_2 \geq \cdots \geq a_n$; $b_1 \geq b_2 \geq \cdots \geq b_n$ be two nonincreasing sequences of real numbers. Then*

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \geq \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}$$

Theorem 8 (Schur) *Let a, b, c be nonnegative reals and $r > 0$. Then*

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

There exist weighted forms of Jensen, AM-GM, and power-mean. We suggest a program for deducing them here, but the details are left as an exercise to the reader. First, obtain integer-weighted forms by setting different combinations of variables equal to one another. Then show that rationally-weighted forms are implied by the integer-weighted forms. Finally, argue by continuity that the weights can be arbitrary reals.

1 Dumbassing: Expansion, AM-GM, and Schur

Dumbassing almost always provides a way to solve homogenous, symmetric inequalities of rational functions. By multiplying by the least common denominator of the fractions at hand one obtains symmetric polynomial inequality, which is then verified by AM-GM (effectively Muirhead's inequality) and Schur's inequality. Situations where it is less likely to be successful include instances of nontrivial equality cases, especially sharp inequalities.

Because Olympiad problems are intended to be clever, they usually have nice solutions. As its colloquial name suggests, dumbassing isn't clever. Rather, it replaces the need for cleverness with sheer calculation. At the 2005 IMO, five members of the USA team solved problem #3, a three variable symmetric inequality of rational functions, more than any other country. Four of those five solvers used dumbassing. Dissatisfaction with the problem has presumably caused such inequalities to fall out of favor at the IMO. However, when working problems of any type, one should first look for easy solutions. For inequalities, this technique should always be *considered,* though not necessarily executed. It has the noteworthy advantages of taking a relatively predictable amount of time to execute and no chance of falsifying an inequality.

(Iran 1996) Show that for all positive reals a, b, c,

$$(ab + bc + ca)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq \frac{9}{4}$$

Solution. Clear the denominators. On the left we obtain

$$\sum_{cyc} 4(a+b)^2(a+c)^2(ab+bc+ca)$$

$$\begin{aligned}
&= 4\sum_{cyc}(a^2+2ab+b^2)(a^2+2ac+c^2)(ab+bc+ca) \\
&= 4\sum_{cyc}(a^4+2a^3b+a^2b^2+2a^3c+4a^2bc+2ab^2c+a^2c^2+2abc^2+b^2c^2)(ab+bc+ca) \\
&= 4\sum_{cyc}\big[\big(a^5b+2a^4b^2+a^3b^3+2a^4bc+4a^3b^2c+2a^2b^3c+a^3bc^2+2a^2b^2c^2+ab^3c^2\big) \\
&\quad+\big(a^4bc+2a^3b^2c+a^2b^3c+2a^3bc^2+4a^2b^2c^2+2ab^3c^2+a^2bc^3+2ab^2c^3+b^3c^3\big) \\
&\quad+\big(a^5c+2a^4bc+a^3b^2c+2a^4c^2+4a^3bc^2+2a^2b^2c^2+a^3c^3+2a^2bc^3+ab^2c^3\big)\big] \\
&= \sum_{sym} 4a^5b+8a^4b^2+10a^4bc+6a^3b^3+52a^3b^2c+16a^2b^2c^2,
\end{aligned}$$

and on the right we obtain

$$\begin{aligned}
& 9(a+b)^2(b+c)^2(c+a)^2 \\
&= 9\left(2abc+\sum_{sym}a^2b\right)^2 \\
&= 9\left(4a^2b^2c^2+\sum_{sym}\left[4a^3b^2c+a^2b(a^2b+a^2c+ab^2+b^2c+ac^2+bc^2)\right]\right) \\
&= 9\left(4a^2b^2c^2+\sum_{sym}\left[4a^3b^2c+a^4b^2+a^4bc+a^3b^3+a^2b^3c+a^3bc^2+a^2b^2c^2\right]\right) \\
&= \sum_{sym} 9a^4b^2+9a^4bc+9a^3b^3+54a^3b^2c+15a^2b^2c^2
\end{aligned}$$

The desired is

$$LHS-RHS=\sum_{sym}4a^5b-a^4b^2+a^4bc-3a^3b^3-2a^3b^2c+a^2b^2c^2\geq 0$$

This is true since

$$\begin{aligned}
\sum_{sym}4a^5b-a^4b^2-3a^3b^3 &\geq 0 \\
\sum_{sym}a^4bc-2a^3b^2c+a^2b^2c^2=abc\sum_{sym}\left(a^3-a^2b+abc\right) &\geq 0,
\end{aligned}$$

by Muirhead and Schur's inequality, respectively. □

Various shorthands exist to expedite the expansion process, but to avoid digression, we don't elaborate on them here.

2 Hölder

Hölder is most useful in inequalities that suggest a factored form. It is also more practical than AM-GM in problems with only cyclic symmetry. Furthermore, applied globally, Hölder is sharper than AM-GM, which is evident upon noting that nontrivial equality cases involving 0 are possible by design. The intended solution for IMO 2005 problem #3 involved Cauchy.

Sometimes the required factorization is introduced by simply multiplying it into both sides of the given inequality. In this case, a good way to detect the required factor or factors is to plug in an arbitrary weights and inspect the outcome for nice possibilities. Another idea is to apply Cauchy locally by applying it to squares expanded under an artificial square root, as is the case in our example.

(Gabriel Dospinescu) Show that for all positive reals a, b, c, x, y, z such that $xy + yz + zx = 3$,

$$\frac{a(y+z)}{b+c} + \frac{b(z+x)}{c+a} + \frac{c(x+y)}{a+b} \geq 3$$

Solution. The desired is equivalent to

$$(x+y+z)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq 3 + \sum_{cyc\{a,x\}} \frac{ax}{b+c}$$

Cauchy's inequality now gives

$$\begin{aligned}
&(x+y+z)\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \\
= \quad & \sqrt{\left([2(xy+yz+zx)] + \sum_{cyc} x^2\right)\left(\left[2\sum_{cyc}\frac{ab}{(c+a)(c+b)}\right] + \sum_{cyc}\frac{a^2}{(b+c)^2}\right)} \\
\geq \quad & \sqrt{[2(xy+yz+zx)]\left[2\sum_{cyc}\frac{ab}{(c+a)(c+b)}\right]} + \sum_{cyc\{a,x\}}\frac{ax}{b+c}
\end{aligned}$$

Since $xy + yz + zx = 3$, we need only show

$$\sum_{cyc} \frac{ab}{(c+a)(c+b)} \geq \frac{3}{4}$$

This is easy, since after clearing denominators we are left with

$$\sum_{sym} 4a^2b \geq \sum_{sym} 3a^2b + abc,$$

which is evident by AM-GM. □

3 Jensen/Karamata (Majorization)

Jensen and Karamata are the basis of most smoothing type solutions. This is because they offer a convenient way of rigorously expressing that we can move variables together or apart. Because they depend only on convex and concave functions, they constitute a particularly natural vehicle for solving inequalities with trigonometry or radicals. Also like Cauchy and Holder, these smoothing inequalities are able to handle equality cases involving zero. (What happens if one of the weights is 0?)

A good general strategy is to plug arbitrary weights into Jensen and inspect for nice choices. Our example shows how Jensen applied to a radical can actually force a cyclic inequality into a fully symmetric form.

(Mock IMO 2005) a, b, c are positive reals. Show that

$$\frac{a}{\sqrt{2a^2+2b^2}} + \frac{b}{\sqrt{2b^2+2c^2}} + \frac{c}{\sqrt{2c^2+2a^2}} \leq \frac{3}{2}$$

Solution. Let f be the concave function $f(x) = \sqrt{x}$. Jensen's inequality now gives

$$\begin{aligned}
\sum_{cyc}\frac{a}{\sqrt{2a^2+2b^2}} \quad &= \quad \sum_{cyc}(2a^2+2c^2) f\left(\frac{a^2}{(2a^2+2c^2)^2(2a^2+2b^2)}\right) \\
&\leq \quad \left[\sum_{cyc}(2a^2+2c^2)\right] f\left(\frac{\sum_{cyc}\frac{a^2}{(2a^2+2c^2)(2a^2+2b^2)}}{\sum_{cyc} 2a^2+2c^2}\right) \\
&= \quad \sqrt{\sum_{cyc}\frac{a^2(a^2+b^2+c^2)}{(a^2+c^2)(a^2+b^2)}}
\end{aligned}$$

Thus, it suffices to show that for positive reals x, y, z,

$$\sum_{cyc} \frac{x(x+y+z)}{(x+y)(x+z)} \leq \frac{9}{4}$$

After clearing denominators this reduces to

$$\sum_{sym} 8a^2b + 4abc \leq \sum_{sym} 9a^2b + 3abc,$$

and is evident by AM-GM. $\square$

4 Problems

1. (MOP lore) Show that for positive reals a, b, c,

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}.$$

2. (IMO 95/2) Show that for all positive reals a, b, c with product 1,

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

3. (MOP 04) Show that for all positive reals a, b, c,

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \geq 3$$

4. Let a, b, c be positive reals with product 1. Show that

$$\frac{2}{(a+1)^2+b^2+1} + \frac{2}{(b+1)^2+c^2+1} + \frac{2}{(c+1)^2+a^2+1} \leq 1$$

5. (Ukraine 01) Let a, b, c, x, y, z be nonnegative reals such that $x + y + z = 1$. Show that

$$ax + by + cz + 2\sqrt{(ab+bc+ca)(xy+yz+zx)} \leq a+b+c$$

6. (Vascile Cartoaje) Let $p \geq 2$ be a real number. Show that for all nonnegative reals a, b, c,

$$\sqrt[3]{\frac{a^3+pabc}{1+p}} + \sqrt[3]{\frac{b^3+pabc}{1+p}} + \sqrt[3]{\frac{c^3+pabc}{1+p}} \leq a+b+c$$

7. (Aaron Pixton) Let a, b, c be positive reals with product 1. Show that

$$5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq (1+a)(1+b)(1+c)$$

8. (IMO 01/2) Let a, b, c be positive reals. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

9. Show that for all positive reals a, b, c,

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

10. (USAMO 04/5) Let a, b, c be positive reals. Prove that

$$\left(a^5 - a^2 + 3\right)\left(b^5 - b^2 + 3\right)\left(c^5 - c^2 + 3\right) \geq (a+b+c)^3$$

11. (Titu Andreescu) Show that for all nonzero reals a, b, c,

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

12. (IMO 96 Shortlist) Let a, b, c be positive reals with $abc = 1$. Show that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

13. Let a, b, c be positive reals such that $a + b + c = 1$. Prove that

$$\sqrt{ab+c} + \sqrt{bc+a} + \sqrt{ca+b} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$$

14. (APMO 2005/2) Let a, b, c be positive reals with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(a^3+1)(b^3+1)}} + \frac{b^2}{\sqrt{(b^3+1)(c^3+1)}} + \frac{c^2}{\sqrt{(c^3+1)(a^3+1)}} \geq \frac{4}{3}$$

15. Let a, b, c be real numbers such that $abc = -1$. Show that

$$a^4 + b^4 + c^4 + 3(a+b+c) \geq \frac{a^2}{b} + \frac{a^2}{c} + \frac{b^2}{c} + \frac{b^2}{a} + \frac{c^2}{a} + \frac{c^2}{b}$$

16. (USAMO 80/5) Show that for all nonnegative reals $a, b, c \leq 1$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1$$

17. (Darij Grinberg) Show that for all positive reals a, b, c,

$$\frac{\sqrt{b+c}}{a} + \frac{\sqrt{c+a}}{b} + \frac{\sqrt{a+b}}{c} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}$$

18. Show that for all positive reals a, b, c,

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq a + b + c$$

19. (IMO 05/3) Prove that for all positive a, b, c with product at least 1,

$$\frac{a^5 - a^2}{a^5 + b^2 + c^2} + \frac{b^5 - b^2}{b^5 + c^2 + a^2} + \frac{c^5 - c^2}{c^5 + a^2 + b^2} \geq 0$$

20. Let a, b, c be reals with $a + b + c = 1$ and $a, b, c \geq -\frac{3}{4}$. Prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}$$

21. Let a, b, c, x, y, z be real numbers such that

$$(a+b+c)(x+y+z) = 3, \quad (a^2+b^2+c^2)(x^2+y^2+z^2) = 4$$

Prove that

$$ax + by + cz \geq 0$$

22. (Po-Ru Loh) Let a, b, c be reals with $a, b, c > 1$ such that

$$\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1$$

Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1$$

23. (Arqady) Show that for positive reals a, b, c with $a^3 + b^3 + c^3 = 3$,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2}$$

5 Homework

1. Show that the majorization inequality implies Jensen's inequality.
2. Show that the AM-GM inequality follows from Jensen's inequality.
3. Prove Hölder's inequalty using only AM-GM.
4. Use Hölder's inequality to prove the power-mean inequality.
5. Prove the rearrangement inequality from first principles.
6. Show that Chebyshev's inequality is implied by rearrangement.
7. Use rearrangement to prove Schur's inequality.
8. (Technical) Prove the majorization inequality using the definition of a convex function.
9. Prove the weighted forms of Jensen's inequality, AM-GM, and the power-mean inequality. First, suppose f is a convex function on an interval I and $x_1, \ldots, x_n \in I$. Then for nonnegative reals $w_1, \ldots, w_n$ with a postive sum,

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq (w_1 + \cdots + w_n) f\left(\frac{w_1 x_1 + \cdots + w_n x_n}{w_1 + \cdots + w_n}\right).$$

Second, if $a_1, \ldots, a_n$ are postive reals,

$$\frac{w_1 a_1 + \cdots + w_n a_n}{w_1 + \cdots + w_n} \geq (a_1^{w_1} \cdots a_n^{w_n})^{\frac{1}{w_1 + \cdots + w_n}}.$$

Third, if r and s are nonzero reals with $r > s$, then

$$\left(\frac{w_1 a_1^r + \cdots + w_n a_n^r}{w_1 + \cdots + w_n}\right)^{1/r} \geq \left(\frac{w_1 a_1^s + \cdots + w_n a_n^s}{w_1 + \cdots + w_n}\right)^{1/s}.$$

When does equality hold?